Inversion and Resolution

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Inversion Scheme

A nonlinear inverse problem is generally solved by iteratively minimizing the discrepancy between data $\mathbf{d}$ and the model response $\mathbf{f}(\mathbf{m})$, normalized by the standard deviations ϵ_i of the data

$$\sum_i \left| \frac{d_i - f_i(\mathbf{m})}{\epsilon_i} \right|^2 = \left\| \mathbf{D}(\mathbf{d} - \mathbf{f}(\mathbf{m})) \right\|_2^2 \to \min \quad \text{with} \quad \mathbf{D} = \operatorname{diag}(\epsilon_i^{-1}). \tag{1}$$

Multi-dimensional problems are generally ill-posed considering data errors. Therefore, one has to introduce regularizing constraints like smoothness (Constable, Parker, & Constable, 1987) or a-priori-information (Jackson, 1979). This can be accomplished by additionally minimizing a semi-norm $\|\mathbf{C}(\mathbf{m} - \mathbf{m}^0)\|$, weighted by a regularization parameter λ

$$\left\| \mathbf{d} - \mathbf{f}(\mathbf{m}) \right\|_2^2 + \lambda \left\| \mathbf{C}(\mathbf{m} - \mathbf{m}_0) \right\|_2^2 \to \min \quad . \tag{2}$$

The matrix $\mathbf{C}$ represents the expectations to the model, e.g., smoothness constraints. $\mathbf{m}_0$ is the reference or a-priori model. The application of the Gauss-Newton method leads to an iterative scheme $\mathbf{m}_{k+1} = \mathbf{m}_k + \Delta\mathbf{m}_k$ solving the regularized normal equations

$$\left((\mathbf{DS})^T \mathbf{DS} + \lambda \mathbf{C}^T \mathbf{C} \right) \cdot \Delta\mathbf{m}_k = (\mathbf{DS})^T \mathbf{D} \left(\mathbf{d} - \mathbf{f}(\mathbf{m}_k) \right) - \lambda \mathbf{C}^T \mathbf{C} (\mathbf{m}_k - \mathbf{m}_0) \quad . \tag{3}$$

Note, that for a local regularization scheme effecting the model update $\Delta\mathbf{m}$ instead of the model $\mathbf{m}$ the latter term vanishes. The Jacobian or sensitivity matrix $\mathbf{S}$ contains the partial derivatives of the model response with respect to the model parameters

$$S_{ij} = \frac{\partial f_i(\mathbf{m})}{\partial m_j} \quad .$$

With $\hat{\mathbf{S}} = \mathbf{DS}$ the equation (3) can be written using generalized inverse matrices $\hat{\mathbf{S}}^\dagger$ and $\mathbf{C}^\dagger$

$$\Delta\mathbf{m} = \hat{\mathbf{S}}^\dagger \mathbf{D} \Delta\mathbf{d} - \mathbf{C}^\dagger \mathbf{C} (\mathbf{m}_k - \mathbf{m}_0) \quad \text{with} \tag{4}$$

$$\hat{\mathbf{S}}^\dagger = (\hat{\mathbf{S}}^T \hat{\mathbf{S}} + \lambda \mathbf{C}^T \mathbf{C})^{-1} \hat{\mathbf{S}}^T \quad \text{and} \quad \mathbf{C}^\dagger = \lambda (\hat{\mathbf{S}}^T \hat{\mathbf{S}} + \lambda \mathbf{C}^T \mathbf{C})^{-1} \mathbf{C}^T \quad .$$

Note, that $\hat{\mathbf{S}}^\dagger \hat{\mathbf{S}} + \mathbf{C}^\dagger \mathbf{C} = \mathbf{I}$.

The Model Resolution

The data are superposed by the response of the true model $\mathbf{m}_{true}$ and the noise $\mathbf{n}$

$$\mathbf{d} = \mathbf{f}(\mathbf{m}_{true}) + \mathbf{n} \quad . \tag{5}$$

Assuming in the k^{th} iteration the model $\mathbf{m}_k$ is already close to the true model, a linearized Taylor expansion of $\mathbf{f}(\mathbf{m}_k)$ yields

$$\mathbf{d} = \mathbf{f}(\mathbf{m}_{true}) + \mathbf{n} = \mathbf{f}(\mathbf{m}_k) + \mathbf{S}(\mathbf{m}_{true} - \mathbf{m}_k) + \mathbf{n} \quad . \tag{6}$$

By insertion of $\mathbf{d} - \mathbf{f}(\mathbf{m}_k)$ from equation (6) into equation (4) we obtain for $\mathbf{m}_{est} = \mathbf{m}_{k+1}$

$$\begin{aligned} \mathbf{m}_{est} &= \mathbf{m}_k + \hat{\mathbf{S}}^\dagger \mathbf{S} \mathbf{D}(\mathbf{m}_{true} - \mathbf{m}_k) - \mathbf{C}^\dagger \mathbf{C}(\mathbf{m}_k - \mathbf{m}_0) \quad + \hat{\mathbf{S}}^\dagger \mathbf{D} \mathbf{n} \\ &= \mathbf{m}_k + \hat{\mathbf{S}}^\dagger \hat{\mathbf{S}} \mathbf{m}_{true} - (\hat{\mathbf{S}}^\dagger \hat{\mathbf{S}} + \mathbf{C}^\dagger \mathbf{C})\mathbf{m}_k + \mathbf{C}^\dagger \mathbf{C} \mathbf{m}_0 \quad + \hat{\mathbf{S}}^\dagger \mathbf{D} \mathbf{n} \\ &= \mathbf{R}^M \mathbf{m}_{true} + (\mathbf{I} - \mathbf{R}^M)\mathbf{m}_0 \quad + \hat{\mathbf{S}}^\dagger \mathbf{D} \mathbf{n} \quad . \end{aligned} \tag{7}$$

The model estimate $\mathbf{m}_{est}$ is constructed of the true model, the starting model and noise artifacts. The matrix $\mathbf{R}^M = \hat{\mathbf{S}}^\dagger \hat{\mathbf{S}}$ combining the procedures of measurement and inversion is called resolution matrix. It serves as a kernel function transferring the reality into our model estimate and can be calculated using the generalized singular value decomposition (Friedel, 2003). Alternatively, the model resolution can be approximated by conjugate gradient techniques (Alumbaugh & Newman, 2000).

The individual columns of $\mathbf{R}^M$ can be plotted like the model vector and display, how an anomaly in the respecting model cell is imaged by the combined process of measurement and inversion. For example, the element R^M_{ij} reveals, how much of the anomaly in the j^{th} model cell is transformed into the i^{th} model cell. Figure 1 displays the model cell resolutions for 4 selected parameters of a 2D profile.

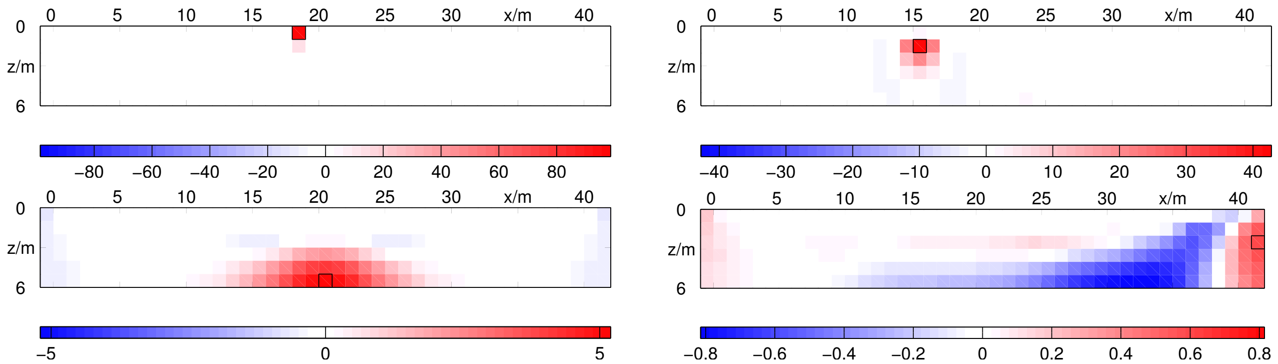

Figure 1: Model cell resolutions (in %) for 4 selected parameters, the cells are marked by black rectangles

The resolution equation (7) is directly linked to the idea of Oldenburg and Li (1999) defining the depth-of-interest (DOI) index. It reveals, to what degree the model parameters are determined by the starting model. The diagonal element R_{ii}^M states, how much of the information is saved in the model estimate and can be interpreted as resolvability of m_i. In Figure 2 the distribution of the R_{ii}^M corresponding to Figure 1 is shown.

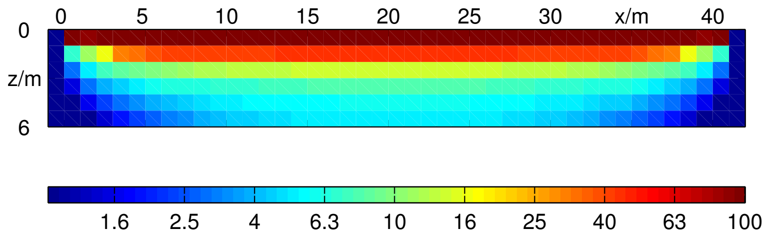

Figure 2: Model resolution of the individual model cells (in %)

Adding up all the diagonal elements we obtain a total information of the inverse process, the information content

$$IC = \sum_i^M \mathbf{R}_{ii}^M \quad . \tag{8}$$

Dividing the information content by the number of data N or model parameter M we obtain the information efficiency IE or the resolution degree RD, respectively.

$$IE = \frac{IC}{N} \quad \text{and} \quad RD = \frac{IC}{M} \quad . \tag{9}$$

References

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