Tree tomography

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Impedance tomography

The problem of electrical impedance tomography is characterized by the continuity equation for the electric potential u

$$\nabla \cdot (\sigma \nabla u) = 0 \tag{1}$$

in a bounded domain Ω, on whose boundary Γ the condition

$$\sigma \frac{\partial u}{\partial n} = j \tag{2}$$

holds. The task is to reconstruct the conductivity σ from measured boundary voltages for a series of applied current functions j. It thus represents an inverse parameter problem on an elliptic partial differential equation. Instead of the conductivity one often employs the reciprocal $\rho = 1/\sigma$, the resistivity. Analog to rocks, it can vary over several decades of magnitude within trees.

Typically one employs a preloaded chain of point-wise assumed electrodes. A current is injected by two adjacent electrodes and voltages are measured on the possible other electrode pairs. This method corresponds to the dipole-dipole method in geoelectrics. Of course other configurations are possible and recommendable, indeed the dipole measurements show superior resolution properties. A measured impedance, which is the voltage Δu per current strength I, is transformed into an apparent resistivity ρ^ausing a configuration factor k

$$\rho^a = k \frac{\Delta u}{I} \quad .$$

Its advantage is not only the comparability of different configurations. Since data and model obtain the same physical unit, the inverse problem can be easier handled. The inversion problem is non-linear. Therefor the model is changes iteratively until the measured data can be accounted. The domain is sub-divided into elements of constant resistivity. The size of the elements should comply with the physical resolution and must not be nonsensically small.

Forward calculation

A fundamental module of the inversion is the simulation of potentials for a given parameter distribution. Since except for a few special cases there is no analytical solution in general we

solve (1) approximatively with the method of finite differences (FD) or finite elements (FE). The main problem are the point-like electrodes which cause an infinite current density at the electrodes. At this position the potential decreases with $1/r$-behavior and can only be approximated well by a mesh which is highly refined.

A remedy is the singularity removal technique. The potential is decomposed into a singular, but known, part u_p, the primary potential, and the unknown, but regular, secondary potential u_s.

For u_s the partial differential equation

$$\nabla \cdot (\sigma \nabla u_s) = \nabla \cdot ((\sigma_0 - \sigma) \nabla u_p \tag{3}$$

holds with the boundary condition

$$\sigma \frac{\partial u_s}{\partial n} = (\sigma_0 - \sigma) \frac{\partial u_p}{\partial n} \quad . \tag{4}$$

σ_0 is a homogeneous conductivity distribution with the value at the electrode. For exact circle geometry we can calculate u_p analytically. Since the singular current density was removed we can solve for u_s on a moderate mesh which saves a lot of computing time. When the boundary departs from the circle form u_p is not known. However, since the calculation of u_s has to be carried out many times it the determination of u_p is put into perspective.

Thus the triple-grid technique works: A relatively coarse parameter mesh defines the cells whose resistivities are to be determined. The forward calculation is accomplished on a finer, globally refined secondary field mesh in every iteration. On the locally highly refined primary field mesh the primary potential is calculated once in the beginning of the inversion. As a byproduct we obtain the topography effect. It reveals, how the tree shape affects without consideration of inhomogeneities. So the true configuration factors are determined.

Inversion

For quantification of the inverse problem we define the model vector $\mathbf{m}$. The individual elements m_j characterize the cell resistivities ρ_j. In order to avoid negative resistivities often their logarithm is chosen $m_j = \log \rho_j$. Similarly we proceed with the measurements The data vector contains the logarithms of the apparent resistivities $d_i = \log \rho_i^a$. To each data point an error value ϵ_i is attributed which has either been measured or estimated.

The inversion is bases on the error-weighted least squares minimization of the discrepancy between data and forward response $\mathbf{f}$.

$$\Phi_d = \sum_{i=1}^{N} \frac{d_i - f_i(\mathbf{m})}{\epsilon_i} = \|\mathbf{D}(\mathbf{d} - \mathbf{f}(\mathbf{m}))\|_2^2 \rightarrow \min \quad . \tag{5}$$

The matrix $\mathbf{D}$ contains the reciprocals of the error values on the main diagonal. The minimization of Φ_d is an incorrectly posed problems. Actually there exist a variety of models satisfying the data within error bounds. We regularizing the problem by introducing an additional functional of constraints

$$\Phi_m = \|\mathbf{C}(\mathbf{m} - \mathbf{m}^0)\|_2^2 \quad , \tag{6}$$

the matrix $\mathbf{C}$ may represent a discrete derivation operator such that smooth models are privileged. We minimize the linear combination $\Phi = \Phi_d + \lambda\Phi_m$. The regularization parameter λ is the most important quantity in inversion. It determines the strength of the constraints, thus the smoothness of the model. The value can be adjusted manually with experience. However there exist methods for optimized selection as the L-curve criterion, but they are not guaranteed to work properly. The crucial point is if the data are fit within the error bound, thus not over-accurately. We observe the chi-squared misfit $\chi^2 = \Phi_d/N$. The optimal value is 1, however for practical purposes values below 10 are suggestive.

In every iteration the a new model $\mathbf{m}^{k+1}$ is calculated from the preceding model $\mathbf{m}^k$

$$\mathbf{m}^{k+1} = \mathbf{m}^k + \tau^k \Delta\mathbf{m}^k \quad .$$

The step length τ_k is determined by a line search algorithm such that Φ_d is minimized over $0 \leq \tau \leq 1$. We apply an inexact line search which interpolates the function values between $\mathbf{f}(\mathbf{m}^k)$ and $\mathbf{f}(\mathbf{m}^{k+1})$ in order to save computing time. Therefore at most two forward calculations are enforced in one iteration.

The application of the Gauss-Newton method to minimize Φ requires the solution of

$$\left(\mathbf{S}^T\mathbf{D}^T\mathbf{D}\mathbf{S} + \lambda\mathbf{C}^T\mathbf{C}\right) \cdot \Delta\mathbf{m}^k = \mathbf{S}^T\mathbf{D}^T\mathbf{D}\left(\mathbf{d} - \mathbf{f}(\mathbf{m}^k)\right) - \lambda\mathbf{C}^T\mathbf{C}(\mathbf{m}^k - \mathbf{m}^0) \quad , \qquad (7)$$

which is done with CGLS-based methods.

The sensitivity matrix $\mathbf{S}$ contains the derivatives of the individual forward responses with respect to the model parameters

$$S_{ij} = \frac{\partial f_i(\mathbf{m})}{\partial m_j} \quad .$$

We calculate the entries on the secondary mesh utilizing the reciprocity principle.

Robust Methods

Measurements with many data often show single erroneous values or outliers. In a least squares (L_2) minimization of the discrepancy they can obtain a strong effect onto the result and prevent an appropriate data fit. An L_1 optimization minimizing the sum of the absolute values is much more robust against single outliers but technically complicating. The way out is the concept of iteratively reweighed least squares („robust inversion“). By a proportionate weighting function the data weights are changed such that L_1 behavior is simulated in L_2 minimization. Larger errors are assigned to the data that cannot be fitted and their importance decreases. Consequence is an improved data fit in χ^2 sense. However it may happen that real anomalies are underestimated.

The same proceeding can also be applied on model side. At the edges of high contrasts the corresponding weighting function is lowered. Thus sharp contrasts are enhanced that often comply with the nature of rocks or trees. Indeed the regularization may vanish locally and absurd resistivity contrasts arise.

A restriction of resistivity values can be applied by logarithmic barriers: Instead of $\log m_j$ we choose the m_j to be the logarithm of the difference $\log(\rho_j - \rho_l)$ or $m_j = \log(\rho_u - \rho_j)$. Then ρ_l and ρ_u represent a lower or upper resistivity barrier, respectively. Also a combination of both is possible.

A synthetic example

We demonstrate the technique using a synthetic model: Figure 1 shows the resistivity distribution of the model.

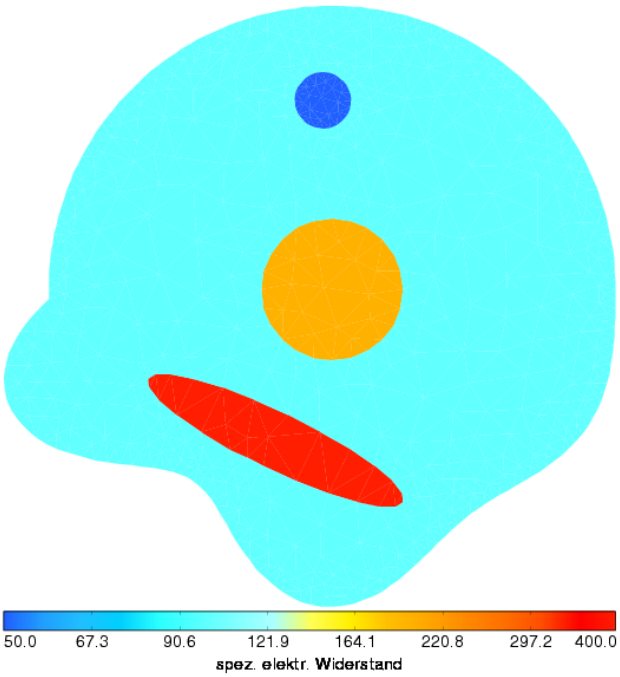

Abbildung 1: The synthetic model

In a homogeneous area of 100 Ωm two resistive (200 and 500 Ωm) and a conductive body (50 Ωm) are placed. Whereas the upper half of the tree has circle geometry, the lower half of the model shows definite undulations which are not uncommon. In Figure 2 we see on the left hand side the synthetic dipole-dipole data that have been transformed with the analytical configuration factor of circles.

Whereas at the bulges of the tree appear increased resistivities the concavities are associated to decreased data. By the simulation on the primary mesh the topography effect (center) is calculated. It shows similar characteristics as the raw data. We utilize the values to calculate the real configuration factors. The right image shows the so corrected data. Clearly the artifacts caused by the tree shape vanish and resistive anomalies are visible that do apparently correspond to the synthetic bodies.

Figure 3 shows the inversion results for three different values of the regularization parameter λ. The large value of $\lambda = 30$ (left) generates a relatively smooth model. It shows the essential structures but lacks resistivity contrasts. In the middle image ($\lambda = 3$) the contrasts are more distinct. One clearly sees the three bodies, even though the resistive ones cannot be separated completely. However the choice of a too small regularization strength may lead to artifacts that do not correspond to the resolution properties of the measurement. The optimum value is for most data sets between 10 and 30. Excellent data sets may be better explained with values below 10, whereas data with bad quality may require regularization strengths of several hundreds.

Figure 4 shows the result of the inversion with „blocky model“ option. It can reproduce

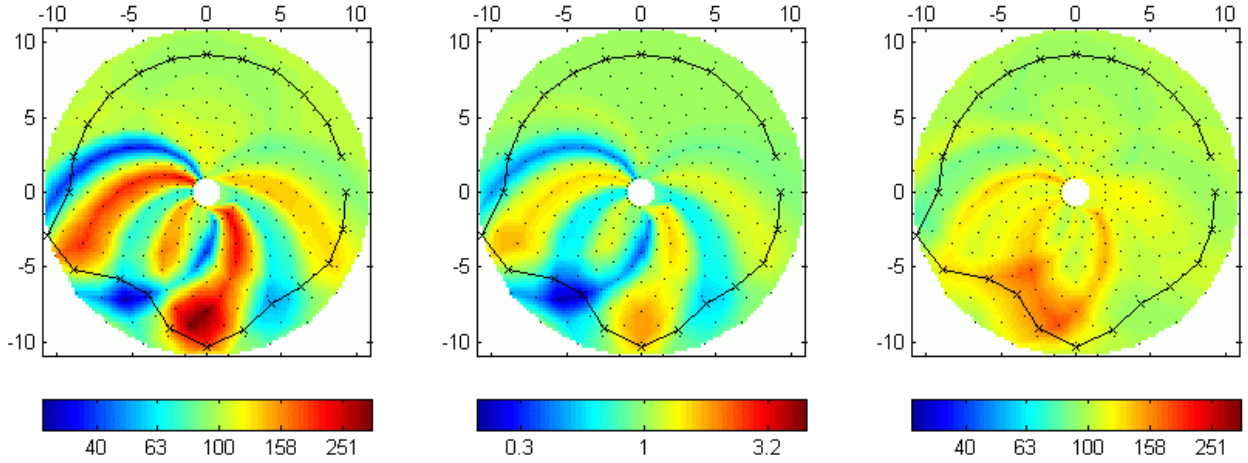

Abbildung 2: Raw data (left), topography effect (center) and corrected data (right) for the synthetic model

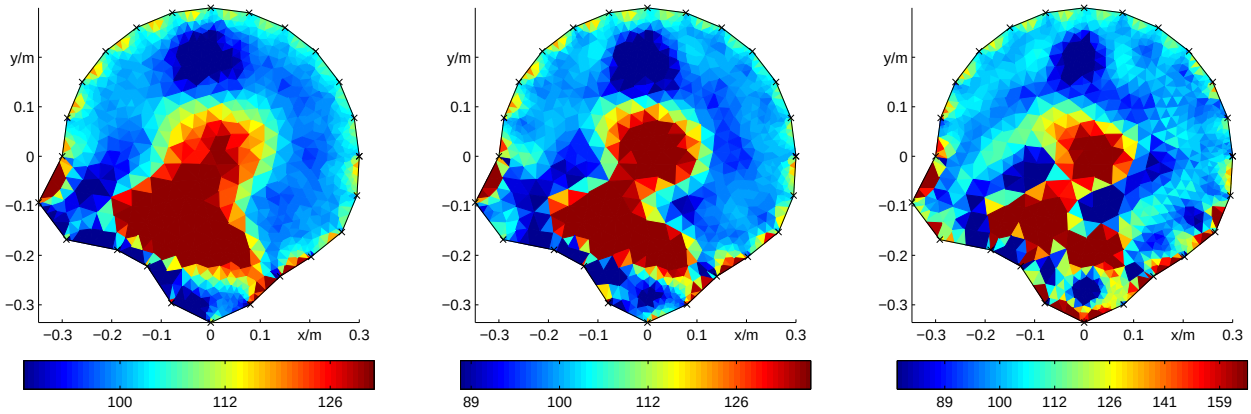

Abbildung 3: inversion results for value of λ =30 (left), 3 (center) and 0.3 (right)

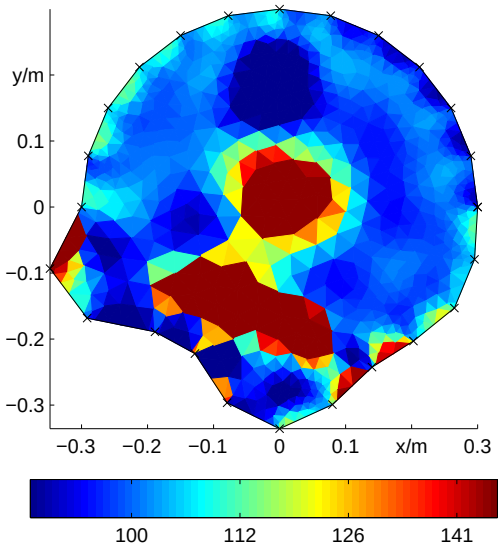

Abbildung 4: Inversion result with the „blocky model“ option

the synthetic model best. In all cases artifacts arise directly at the boundary that can traced back to insufficient discretization. However they are rarely of practical importance for the interpretation and can thus be neglected.

Field examples

will follow

Induced polarization

will follow too