An algorithm for refraction tomography

Introduction

In Geophysics refraction of seismic waves is used to detect fast subsurface layers, since from a certain offset the refracted wave arrives earlier than the ground wave. From the slope of the travel time curve we can determine the velocities. Sharp bends give allow for calculating the depth of layer boundaries.

The seismic velocity v is the partial derivative of the path length l with respect to time t, $v = \partial l/\partial t$. Typical values are some 100 m/s for unconsolidated sediments up to several 1000 m/s for solid rocks. Often the reciprocal, the slowness $s = 1/v$ is considered. In the following we present an algorithm that acquires the velocity as a spatial function, i.e. it enables a tomography of the earth on the basis of seismic travel time data. It does not play a role if Refraction (Ra) data are used (from only one side) or tomography between boreholes or at excavated objects is applied. Note that reflections are not considered.

Refraction tomography has been used since the 80s, e.g., White (1989) provides an overview. They mainly consist of a ray propagation computation and inverse projections such as iterative reconstruction techniques. Zhang et al. (1998) addresses different inversion methods and resolution analysis. Practical details are covered by, e.g., Lanz et al. (1998).

Forward problem

The forward problem is to simulate the ray propagation for a given slowness distribution $s(\vec{r})$ in order to predict the first arrivals. This procedure is often referred to as ray tracing.

The total travel time is the integral over the whole path l

$$t = \int_l \mathrm{d}t = \int_l \frac{\mathrm{d}l}{v} = \int_L s \; \mathrm{d}l \quad . \tag{1}$$

We subdivide the modeling domain in M model cells of constant[1] slowness s_i and transfer the integral into a sum

$$t = \sum_{i=1}^{M} l_i s_i \quad . \tag{2}$$

Equation (2) can be posed as matrix-vector-product

$$\mathbf{t} = \mathbf{L}\mathbf{s} \quad , \tag{3}$$

where $\mathbf{t}$ is the travel time vector of the N measurements and $\mathbf{s}$ is the vector of slowness values s_i. The matrix $\mathbf{L}$ is referred to as path matrix and contains in the elements L_{ij} the path length of the i^{th} ray through the j^{th} element. Since every ray covers only a few cells, this matrix is generally sparse.

In general there are different methods:

1. With the *ray method* a bundle of rays is tracked using refraction laws and the incident angle is varied until the fastest ray reaches the arrival point. The calculated ray path is exact due to the analytical relations.

[1] The discretization may also be done using gradient models.

2. From the wave equation we derive the *Eikonal equation*

$$|\nabla T(\vec{r})| = s(\vec{r}) \quad \text{bzw.} \quad (\nabla T)^2 = s^2 \quad , \tag{4}$$

for the travel time function T, which has to be solved by appropriate numerical methods, e.g. finite elements. The accuracy of the approximate solution can be arbitrarily improved by refining the mesh.

3. An inexact method is the path method. The rays are restricted to the edges of the mesh and, thus, a weighted Shortest path problem is solved, as it is the case for routing machines. Since not every path can be used, the travel times are always overestimated. However the discrepancy diminishes with increasing refinement.

Initially we use 3. (Shortest Path), because it is easy to implement and agrees with the perception of refraction according to Huygens law. In case of 1. the back refraction from a fast layer is a problem to be solved. A fundamental idea is Dijkstra's algorithm (Dijkstra, 1959). Starting from the shot point a tree of travel times is spanned which spreads on the whole domain. It can be improved by adapted data structured (Fibonacci heap).

The A* algorithm is a generalization of Dijkstra's method for non-positive weights. It is much faster by the introduction of an upper bound (e.g. free air path). However the advantage disappears, since for each geophone the problem has to be resolved, whereas Dijkstra solves the problem for all at once. Therefore we use the latter which is independent on the mesh dimension.

For the calculation appropriate meshes has to be created. Structured meshes show preferred directions and cumber a physically realistic ray propagation. Unstructured meshes have the additional advantage of being able to involve surface topography. With the triangle algorithm of Shewchuk (1996) a 2d mesh is created on the basis of given number of fixed nodes in such a way that the mesh meets quality criteria (preferably uniform triangles).

The generation of three-dimensional meshes is a bigger problem. A large flexibility is obtained by tetrahedral meshes that can be created, e.g., by the free mesh generator TetGen (Si, 2003) according to similar principles.

Inverse problem

Objective is to find a (plausible) velocity distribution that is able to explain the measured travel times, i.e. to find an inverse operator. The way propagation itself is a linear operator. Since the ray path (especially for refraction seismics) depends critically on the velocities, the inverse problem is highly non-linear and has to be solved iteratively. Starting from a model $\mathbf{s}^0$ new models are created successively based on the discrepancy of data and model response, solving

$$\mathbf{s}_{k+1} = \mathbf{s}^k + \Delta\mathbf{s}^k = \mathbf{s}^k + \mathbf{L}^\dagger(\mathbf{s}^k)(\mathbf{t} - \mathbf{L}(\mathbf{s}^k)\mathbf{s}^k) \quad ,$$

then $\mathbf{L}$ is recomputed and so on. $\mathbf{L}^\dagger$ is the used inverse operator.

Classical methods for ray problems are algebraic reconstruction techniques (ART) and the simultaneous iterative reconstruction technique (SIRT). Moreover truncated SVD inversions (tsvd) are used since its resolution properties are easily accessible (White, 1989). With regard to a joint inversion we want to apply a quadratic minimization problem. The function to be

minimized is the sum of the squared discrepancies between data $\mathbf{t}$ and model response $\mathbf{Ls}$

$$\Phi_d = \sum_{i=1}^{N} t_i - (\mathbf{Ls})_i = \|\mathbf{t} - \mathbf{Ls}\|_2^2 \rightarrow \min \quad . \tag{5}$$

The application of the Gauss-Newton method leads to the normal equations

$$\Delta\mathbf{s}^k = (\mathbf{L}^T\mathbf{L})^{-1}\mathbf{L}^T(\mathbf{t} - \mathbf{Ls}^k) \quad .$$

However we find this method to be instable with respect to data errors. Therefore we need to regularize the problem by introducing a model functional Φ_m, which is weighed by a regularization parameter λ

$$\Phi = \Phi_d + \lambda\Phi_m \quad \text{mit} \quad \Phi = \|\mathbf{Cs}\|_2^2 \quad .$$

$\mathbf{C}$ is an operator, whose image has to be minimized. Typical is a derivative enforcing a smooth slowness distribution. The according normal equations are

$$\Delta\mathbf{s}^k = (\mathbf{L}^T\mathbf{L} + \lambda\mathbf{C}^T\mathbf{C})^{-1}(\mathbf{L}^T(\mathbf{t} - \mathbf{Ls}^k) - \lambda\mathbf{C}^T\mathbf{Cs}) \quad . \tag{6}$$

Equation 6 is solved until Φ stagnates. The value $\delta t = \sqrt{\Phi_d/N}$ shows the mean quadratic deviation (MQD). The regularization parameter controls the model smoothness and has to be chose such that δt reflects the measuring errors.

At the beginning of the inversion an appropriate parameterization has to be chose. It is determined by the shot/geophone positions and the desired maximum depth (depends on the used offsets). To enable a change in a certain cell, it must be covered by at least one ray path. In order to get a good coverage we start with a gradually layered model (White, 1989) or a multi-layer model (Lanz et al., 1998). The velocities may be extracted from the common offset stack.

The ray coverage is a first measure to evaluate the model, e.g. uncovered cells may be blanked. More sophisticated studies use the resolution analysis to quantify the reliability of inversion results (Zhang et al., 1998).

The model parameter to be reconstructed may also be the velocity. For this the matrix $\mathbf{L}$ in the inverse problem is multiplied by the inner derivative, $\partial v/\partial s = -1/s^2$, of the individual model cells. Both methods may enhance slow or fast structured. A possible alternative is the use of the logarithm as model parameter. The weighting function is then $-1/s$. In our experience it provides the stablest results which can be proved by observing the singular value spectra.

Application to field data

We want to apply the method to field data. A 50 m long profile in Königssee was obtained to detect the bedrock surface. Figure 1 shows the shot points and their measured travel times color-coded.

The curve slopes decrease with increasing offset showing a fast layer in depth. However, the strong variations do also show that the earth is not one-dimensional. From the variation we get a first concept of measuring errors in a range of milliseconds.

The first shot cuts the abscissa at the same point as the second, the same is the case for the last shot. Since this indicates positioning errors, the first and last shots are neglected in the following.

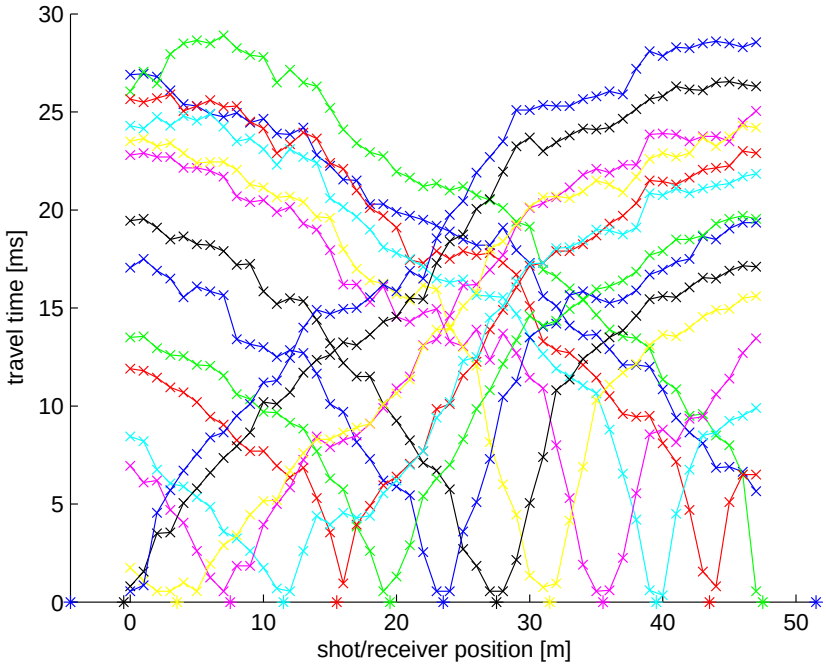

Figure 1: Field data on profile 1:travel times for the individual shot points as function of space, the stars denote the shot locations

Model depth was chosen 15 m, one has to find an optimum together with the velocities. Those were determined by the common-offset stack to be $v_{min} = 500$m/s und $v_{max} = 1500$m/s. According to their midpoints to every triangle a velocity is associated by interpolation show in Fig. 2.

To clarify the the ray propagation the paths for the first shot are plotted. In ideal case (very fine discretization) they show parable shape. Generally this can be recognized. However there are deviations due to the restricted ray paths as well as the discontinuous velocity distribution.

The regularization parameter has been chosen in such a way that the MQD agrees with the estimated error of 1 ms, i.e., the data are fitted within error bounds (discrepancy principle) under the consumption of Gaussian noise. A smaller regularization improves the data fit but shows non-resolvable structures, whereas a stronger regularization cannot fit the data well.

Figure 3 shows the inversion result after 4 iterations. Near the surface, especially in the right part, we see a layer of low velocity (600-800 m/s), which can be interpreted as unconsolidated topsoil. From a depth of approx. 5 m on a layer of horizontally varying high velocity (>2000 m/s) is visible, which we interpret as bedrock. Since cells without ray coverage are not determined by the data, they are blanked. With a few exceptions these are the boundary parts and larger depths giving clue to a better parameterization.

Figure 4 shows the fit of the data (crosses) by the model response (solid line). Most data are met with sufficient quality. However there are parts of the travel time curves, e.g. at their beginning and end, that can't be explained with the permitted complexity. This may be due to higher inhomogeneities at the near-surface layers as well as systematic errors.

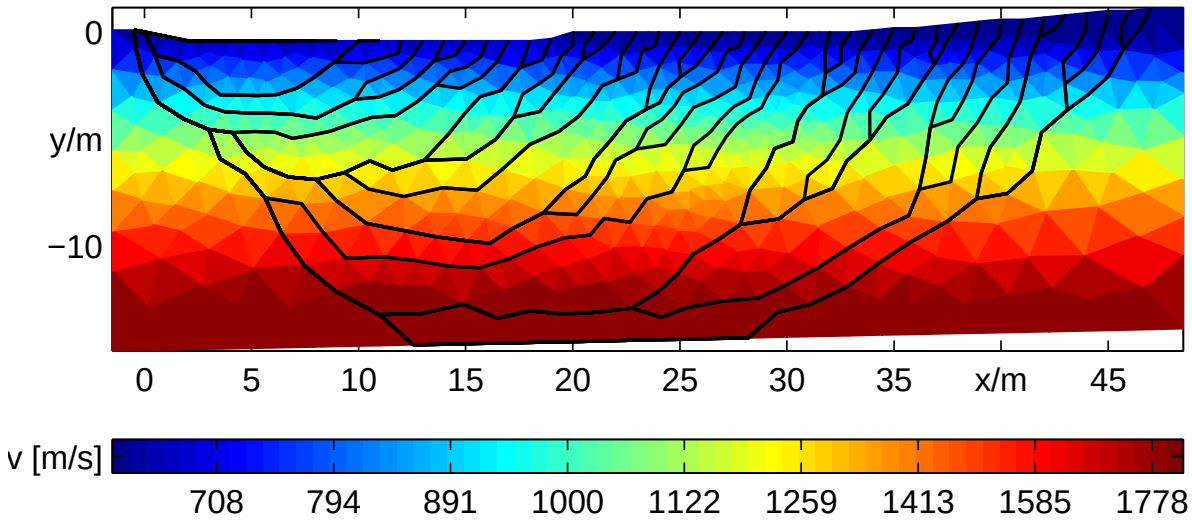

Figure 2: Starting model: velocity distribution and ray paths for first shot

Conclusions

Firstly a refraction tomography can be done with simple means, i.e., Shortest Path forward solver and Gauss-Newton inversion. However the ray paths are not completely realistic. Finer meshes allow for better approximation of ray paths, but may create smearing effects (White, 1989) and do moreover not conform the resolution properties. We expect the bias by the approximate forward solution to occur throughout the model and thus cause no structural deformations. An improvement of the forward solver will be done solving the Eikonal equation.

The inverse problem is, compared to potential methods, similarly ill-posed and has also to be regularized. A sophisticated error-estimation may avoid over- or underfitting the data. The non-linearity and thus the dependence from a starting model is much stronger. Resolution studies may help to restrict the diversity of models.

A basic principle is the application of different methods to obtain more reliable models. Ideally this will be done in the framework of a joint inversion, e.g. with dc resistivity data, since this works on similar depth ranges. Special attention has to be payed to the parameterization and the use of identical meshes or appropriate interpolation algorithms.

References

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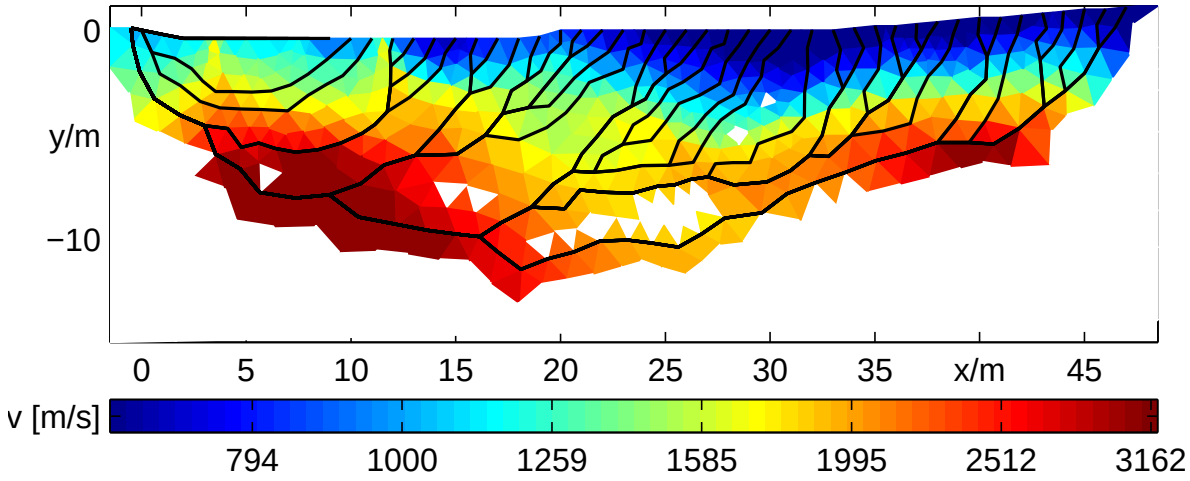

Figure 3: Inversion result: velocity distribution with ray paths for first shot point

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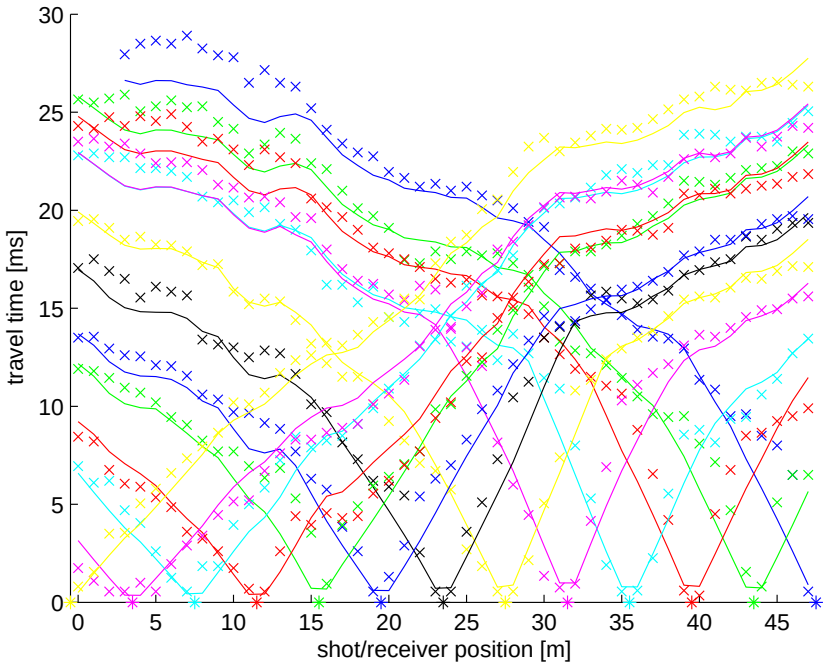

Figure 4: Comparison of measured (crosses) data and model response (lines)